

A KIND OF LOCAL STRONG ONE-SIDE POROSITY

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Abstract. We define and study the completely strongly porous at 0 subsets of $[0, \infty)$. Several characterizations of these subsets are obtained, among them the description via an universal property and structural one.

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1 Introduction

The basic ideas concerning the notion of set porosity for the first time appeared in some early works of Denjoy [5], [6] and Khintchine [13] and then were arisen independently in the study of cluster sets in 1967 (Dolženko [7]). Denjoy was interested in obtaining a classification of perfect sets on the real line in terms of the relative sizes of the complementary intervals. Khintchine had required a convenient way of describing certain arguments that use density considerations. The notion of a set of σ -porosity was defined by E. P. Dolženko [7]. The basic structure of porous sets and σ -porous sets has been studied in [9], [10] and [21]. A useful collection of facts related to the notion of porosity can be found in [20]. A number of theorems exists in the theory of cluster sets which use the notion of σ -porosity (see, for example, [24], [25], [26], [27]). No less important is a question about relationship between porosity and dimension. In many applications the information on the dimension of certain sets is obtained via porosity. See the use of porosity, for example, in connection with free boundaries [11] and complex dynamics [18]. Estimates of dimension in terms of porosity were obtained for a wide variety of notions of porosity (and dimension) in [2], [8], [14], [15], [16], [17], [19], etc. The porosity (in an appropriate sense) of many natural sets and measures was investigated in [2], [4], [14], [22]. Moreover, the relationship between porosity and other geometric concepts such as conical densities and singular integrals was explored in [4], [12], [16]. Porosity is also a property which is preserved, for example, under quasisymmetric maps [23]. Thereby the notion of set porosity plays an implicit role in different questions of analysis.

Many nontrivial modifications of the notion of porosity are used at present. The comparison of different definitions, and a survey of results can be found in [28]. Our paper is also a contribution to this line of studies and we introduce a new subclass of strongly porous at 0 subsets of $\mathbb{R}^+ = [0, +\infty)$.

Let us recall the definition of the right porosity. Let E be a subset of \mathbb{R}^+ .

Definition 1.1. *The right porosity of E at 0 is the quantity*

$$p^+(E, 0) := \limsup_{h \rightarrow 0^+} \frac{\lambda(E, 0, h)}{h} \quad (1.1)$$

where $\lambda(E, 0, h)$ is the length of the largest open subinterval of $(0, h)$ that contains no point of E . The set E is strongly porous on the right at 0 if $p^+(E, 0) = 1$.

Let $\tilde{\tau} = \{\tau_n\}_{n \in \mathbb{N}}$ be a sequence of real numbers. We shall say that $\tilde{\tau}$ is *almost decreasing* if the inequality $\tau_{n+1} \leq \tau_n$ holds for sufficiently large n . Write \tilde{E}_0^d for the set of almost decreasing sequences $\tilde{\tau}$ with $\lim_{n \rightarrow \infty} \tau_n = 0$ and having $\tau_n \in E \setminus \{0\}$ for $n \in \mathbb{N}$.

We use the symbols $ExtE$ and acE to denote the exterior and, respectively, the set of all accumulation points (relative to the space \mathbb{R}^+ with the standard topology) of a set $E \subseteq \mathbb{R}^+$.

Remark 1.2. The set \tilde{E}_0^d is empty if and only if $0 \notin acE$.

Define \tilde{I}_E to be the set of sequences $\{(a_n, b_n)\}_{n \in \mathbb{N}}$ of open intervals $(a_n, b_n) \subseteq \mathbb{R}^+$ meeting the following conditions.

- Every a_n is strictly positive.
- Every interval (a_n, b_n) is a connected component of $ExtE$, i.e., $(a_n, b_n) \cap E = \emptyset$ but for every $(a, b) \supseteq (a_n, b_n)$ we have

$$((a, b) \neq (a_n, b_n)) \Rightarrow ((a, b) \cap E \neq \emptyset).$$

- The limit relations $\lim_{n \rightarrow \infty} a_n = 0$ and $\lim_{n \rightarrow \infty} \frac{b_n - a_n}{b_n} = 1$ hold.

Remark 1.3. $\tilde{I}_E \neq \emptyset$ if and only if $0 \in acE$ and $p^+(E, 0) = 1$.

Define also an equivalence \asymp on the set of sequences of strictly positive numbers as follows. Let $\tilde{a} = \{a_n\}_{n \in \mathbb{N}}$ and $\tilde{\gamma} = \{\gamma_n\}_{n \in \mathbb{N}}$. Then $\tilde{a} \asymp \tilde{\gamma}$ if there are constants $c_1, c_2 > 0$ such that

$$c_1 a_n \leq \gamma_n \leq c_2 a_n \quad (1.2)$$

for sufficiently large $n \in \mathbb{N}$.

Definition 1.4. Let $E \subseteq \mathbb{R}^+$ and $\tilde{\gamma} \in \tilde{E}_0^d$. The set E is $\tilde{\gamma}$ -strongly porous if there is a sequence $\{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E$ such that

$$\tilde{\gamma} \asymp \tilde{a} \quad (1.3)$$

where $\tilde{a} = \{a_n\}_{n \in \mathbb{N}}$. The set E is completely strongly porous (at 0) if E is $\tilde{\gamma}$ -strongly porous for every $\tilde{\gamma} \in \tilde{E}_0^d$.

Remark 1.5. If $0 \notin acE$, then E is completely strongly porous because $\tilde{E}_0^d = \emptyset$.

In what follows the set of all completely strongly porous subsets of \mathbb{R}^+ will be denoted by **CSP**.

The main results of the paper can be informally described by the following way.

- **CSP** - sets are uniformly strongly porous (Theorem 2.18), in the sense that the constants in (1.2) can be chosen independently of $\tilde{\gamma} \in \tilde{E}_0^d$ if $E \in \mathbf{CSP}$.
- If $E \in \mathbf{CSP}$, then there is an universal $\tilde{L} \in \tilde{I}_E$ such that every $\tilde{A} \in \tilde{I}_E$ is a “subsequence” of \tilde{L} (Theorem 2.18).
- A description of the structure of strongly porous on the right at 0 sets $E \subseteq \mathbb{R}^+$ having an universal $\tilde{L} \in \tilde{I}_E$ (Theorem 2.26).
- An explicit design generating all **CSP** - sets (Theorem 3.7).

Olli Martio’s question concerning interconnections between the infinitesimal structure of a metric space (X, d) at a point $p \in X$ and the porosity of the distance set $\{d(x, p) : x \in X\}$ was a starting point in our studies of **CSP** - sets. Some results in this direction can be found in [1] and [3].

2 The *CSP* - sets

We start in our investigations of the *CSP* - sets from the following two examples.

Example 2.1. Let $\{x_n\}_{n \in \mathbb{N}}$ be a strictly decreasing sequence of positive real numbers with $\lim_{n \rightarrow \infty} \frac{x_{n+1}}{x_n} = 0$. Define a set W by the rule

$$(x \in W) \Leftrightarrow (\text{either } x = 0 \text{ or there is } n \in \mathbb{N} \text{ such that } x = x_n).$$

Then W is a closed *CSP* - set and the sequence $\{(x_{n+1}, x_n)\}_{n \in \mathbb{N}}$ belongs to \tilde{I}_W .

Example 2.2. Let $q \in [1, \infty)$ and let W be the set from the previous example. Write

$$W(q) = \bigcup_{x \in W} [x, qx],$$

where $[x, qx]$ is the closed interval with the endpoints x and qx . Then $W(q)$ is a closed *CSP* - set. Let $m_0 \in \mathbb{N}$ be a number such that $qx_{n+1} < x_n$ for every $n \geq m_0$. The sequence $\{(qx_{m_0+n+1}, x_{m_0+n})\}_{n \in \mathbb{N}}$ belongs to $\tilde{I}_{W(q)}$.

Lemma 2.3. Let $E \subseteq \mathbb{R}^+$, $\tilde{\gamma} \in \tilde{E}_0^d$, $\{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E$ and let $\tilde{a} := \{a_n\}_{n \in \mathbb{N}}$. The weak equivalence $\tilde{\gamma} \asymp \tilde{a}$ holds if and only if

$$\limsup_{n \rightarrow \infty} \frac{a_n}{\gamma_n} < \infty \quad (2.1)$$

and

$$\gamma_n \leq a_n \quad (2.2)$$

for sufficiently large n .

Proof. It is easily seen that (2.1) and (2.2) imply $\tilde{\gamma} \asymp \tilde{a}$. Conversely suppose that $\tilde{\gamma} \asymp \tilde{a}$. The membership $\{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E$ yields $\frac{b_n}{a_n} \rightarrow \infty$ with $n \rightarrow \infty$. Since $\tilde{\gamma} \asymp \tilde{a}$, we obtain $\frac{\gamma_n}{a_n} \leq c_2$ for sufficiently large n where c_2 is the constant from (1.2). Consequently there is $N_0 \in \mathbb{N}$ such that the inequality

$$\gamma_n < b_n \quad (2.3)$$

holds if $n \geq N_0$. Since $(a_n, b_n) \cap E = \emptyset$ and $\gamma_n \in E$, inequality (2.3) implies (2.2). To prove (2.1) note that the left inequality in (1.2) is equivalent to

$$\frac{1}{c_1} \geq \frac{a_n}{\gamma_n}$$

where c_1 is the constant from (1.2). Hence (2.1) holds. \square

Corollary 2.4. Let $E \subseteq \mathbb{R}^+$ and let $\tilde{\tau} \in \tilde{E}_0^d$. The set E is $\tilde{\tau}$ -strongly porous if and only if there exists a sequence $\{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E$ such that $\limsup_{n \rightarrow \infty} \frac{a_n}{\tau_n} < \infty$ and $\tau_n \leq a_n$ for sufficiently large n .

The following proposition does not have any applications in the paper but is used in [3] to describe the structure of bounded tangent spaces to general metric spaces.

Proposition 2.5. Let $E \subseteq \mathbb{R}^+$ and let $\tilde{\tau} = \{\tau_n\}_{n \in \mathbb{N}} \in \tilde{E}_0^d$. The following statements are equivalent.

- (i) E is $\tilde{\tau}$ -strongly porous.

- (ii) *There is a constant $k \in (1, \infty)$ such that for every $K \in (k, \infty)$ there exists $N_1(K) \in \mathbb{N}$ such that*

$$(k\tau_n, K\tau_n) \cap E = \emptyset \quad (2.4)$$

if $n \geq N_1(K)$.

Proof. Suppose that E is $\tilde{\tau}$ -strongly porous. By Corollary 2.4 there is a sequence

$$\{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E \quad (2.5)$$

such that $\limsup_{n \rightarrow \infty} \frac{a_n}{\tau_n} < \infty$ and $\tau_n \leq a_n$ for sufficiently large n . Write $k = 1 + \limsup_{n \rightarrow \infty} \frac{a_n}{\tau_n}$, then $\infty > k \geq 2$ and there is $N_0 \in \mathbb{N}$ such that

$$\tau_n \leq a_n < k\tau_n \quad (2.6)$$

for $n \geq N_0$. Let $K \in (k, \infty)$. Membership (2.5) implies the equality $\lim_{n \rightarrow \infty} \frac{b_n}{a_n} = \infty$. The last equality and (2.6) show that there is $N_1 \geq N_0$ such that

$$a_n < k\tau_n < K\tau_n \leq b_n$$

if $n \geq N_1$. Hence the inclusion

$$(k\tau_n, K\tau_n) \subseteq (a_n, b_n) \quad (2.7)$$

holds if $n \geq N_1$. Since

$$E \cap (a_n, b_n) = \emptyset, \quad (2.8)$$

(2.7) implies (2.4). Thus (ii) follows from (i).

Conversely, assume that statement (ii) holds. Let $K > 1$. Then for $K = 2k$ there is $N_0 \in \mathbb{N}$ such that

$$(k\tau_n, 2k\tau_n) \cap E = \emptyset$$

if $n \geq N_0$. Consequently, for every $n \geq N_0$, we can find a connected component (a_n, b_n) of $\text{Ext}E$ meeting the inclusion

$$(k\tau_n, 2k\tau_n) \subseteq (a_n, b_n). \quad (2.9)$$

Write $(a_n, b_n) = (a_{N_0}, b_{N_0})$ for $n < N_0$. Since, for $n \geq N_0$, we have

$$\tau_n \in E, \tau_n < k\tau_n \text{ and } (a_n, k\tau_n) \cap E = \emptyset,$$

the double inequality $\tau_n \leq a_n < k\tau_n$ holds for such n . To prove (i) it is sufficient to show that

$$\{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E.$$

All intervals (a_n, b_n) are connected components of $\text{Ext}E$ and $\lim_{n \rightarrow \infty} a_n = 0$ because $\lim_{n \rightarrow \infty} \tau_n = 0$, so that $\{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E$ if and only if

$$\lim_{n \rightarrow \infty} \frac{b_n}{a_n} = \infty. \quad (2.10)$$

Let K be an arbitrary point of (k, ∞) . Applying (2.4) we can find $N_1(K) \in \mathbb{N}$ such that

$$(k\tau_n, K\tau_n) \subseteq (a_n, b_n)$$

for $n \geq N_1(K)$. Consequently, for such n , we have

$$\frac{b_n}{a_n} \geq \frac{K\tau_n}{k\tau_n} = \frac{K}{k}.$$

Letting $K \rightarrow \infty$ we see that (2.10) follows. \square

It is clear that, if there is $\tilde{\tau} \in \tilde{E}_0^d$ such that E is $\tilde{\tau}$ -strongly porous, then E is strongly porous on the right at 0. Conversely we have the following

Proposition 2.6. *Let $E \subseteq \mathbb{R}^+$ and $0 \in acE$. If E is strongly porous on the right at 0, then there is $\tilde{\tau} \in \tilde{E}_0^d$ for which E is $\tilde{\tau}$ -strongly porous.*

The proof is immediate and can be omitted.

Remark 2.7. If $0 \notin acE$, then E is strongly porous on the right at 0 but there are no $\tilde{\tau} \in \tilde{E}_0^d$ because $\tilde{E}_0^d = \emptyset$.

Definition 2.8. *Let $E \subseteq \mathbb{R}^+$. The set E is uniformly strongly porous (at 0) if there exists a constant $c > 0$ such that for every $\tilde{\tau} \in \tilde{E}_0^d$ there is $\{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E$, satisfying the following conditions:*

- (i) $a_n \geq \tau_n$ for sufficiently large $n \in \mathbb{N}$;
- (ii) the inequality

$$\limsup_{n \rightarrow \infty} \frac{a_n}{\tau_n} \leq c$$

holds.

Remark 2.9. If $0 \notin acE$, then E is uniformly strongly porous since $\tilde{E}_0^d = \emptyset$.

If E is uniformly strongly porous, then $E \in \mathbf{CSP}$. The converse is also true and we prove this in Theorem 2.18 giving below.

Define, for $\tilde{\tau} \in \tilde{E}_0^d$, a subset $\tilde{I}_E(\tilde{\tau})$ of the set \tilde{I}_E by the rule:

$$(\{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E(\tilde{\tau})) \Leftrightarrow (\{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E \text{ and } \tau_n \leq a_n \text{ for sufficiently large } n \in \mathbb{N}).$$

Write

$$C(\tilde{\tau}) := \inf(\limsup_{n \rightarrow \infty} \frac{a_n}{\tau_n}) \quad \text{and} \quad C_E := \sup_{\tilde{\tau} \in \tilde{E}_0^d} C(\tilde{\tau}) \quad (2.11)$$

where the infimum in the left formula is taken over all $\{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E(\tilde{\tau})$.

Proposition 2.10. *Let $E \subseteq \mathbb{R}^+$ and let $0 \in acE$. The set E is strongly porous at 0 if and only if*

$$\tilde{I}_E(\tilde{\tau}) \neq \emptyset \quad (2.12)$$

for every $\tilde{\tau} \in \tilde{E}_0^d$. The set E is completely strongly porous if and only if $C(\tilde{\tau}) < \infty$ for every $\tilde{\tau} \in \tilde{E}_0^d$. The set E is uniformly strongly porous if and only if $C_E < \infty$.

The proof follows directly from definitions 1.1, 1.4, 2.8, Corollary 2.4 and formulas (2.11).

Lemma 2.11. *Let $E \subseteq \mathbb{R}^+$. If $\tilde{\tau} = \{\tau_n\}_{n \in \mathbb{N}} \in \tilde{E}_0^d$ and $\{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E$ are sequences such that $\tilde{a} \asymp \tilde{\tau}$, then \tilde{a} and \tilde{b} are almost decreasing.*

Proof. It suffices to show that \tilde{a} is almost decreasing. If \tilde{a} is not almost decreasing, then there is an infinite $A \subseteq \mathbb{N}$ such that

$$a_{n+1} > a_n \quad (2.13)$$

for every $n \in A$. Since $(a_n, b_n) \cap E = \emptyset$, inequality (2.13) implies that $a_{n+1} \geq b_n > a_n$. By Lemma 2.3 we have $a_n \geq \tau_n$ for sufficiently large n . In addition, for such n , we may suppose also $\tau_n \geq \tau_{n+1}$ because $\tilde{\tau}$ is almost increasing. Consequently, we obtain

$$a_{n+1} \geq b_n > a_n \geq \tau_n \geq \tau_{n+1} \quad (2.14)$$

for sufficiently large $n \in A$. Inequalities (2.14) imply

$$\frac{b_n}{a_n} \leq \frac{a_{n+1}}{\tau_{n+1}}.$$

Hence

$$\infty = \lim_{n \rightarrow \infty, n \in A} \frac{b_n}{a_n} \leq \limsup_{n \rightarrow \infty, n \in A} \frac{a_{n+1}}{\tau_{n+1}} \leq \limsup_{n \rightarrow \infty} \frac{a_{n+1}}{\tau_{n+1}},$$

contrary to Lemma 2.3. \square

Proposition 2.12. *Let $E \subseteq \mathbb{R}^+$, $\tilde{\tau} \in \tilde{E}_0^d$, and let $\{(a_n^{(1)}, b_n^{(1)})\}_{n \in \mathbb{N}}$, $\{(a_n^{(2)}, b_n^{(2)})\}_{n \in \mathbb{N}}$ be two sequences belonging to \tilde{I}_E . If $\tilde{a}^1 \asymp \tilde{\tau}$ and $\tilde{a}^2 \asymp \tilde{\tau}$, where $\tilde{a}^i := \{a_n^{(i)}\}_{n \in \mathbb{N}}$, $i = 1, 2$, then there is $N_0 \in \mathbb{N}$ such that*

$$(a_n^{(2)}, b_n^{(2)}) = (a_n^{(1)}, b_n^{(1)}) \quad (2.15)$$

for every $n \geq N_0$.

Proof. Let us denote by E^1 the closure of E in \mathbb{R}^+ . Using Remark 1.2 we see that $0 \in acE^1$ and $\tilde{\tau} \in \tilde{E}_0^{1d}$. Since the sequences $\{(a_n^{(i)}, b_n^{(i)})\}_{n \in \mathbb{N}}$, $i = 1, 2$, belong to \tilde{I}_E , they also belong to \tilde{I}_{E^1} . By Lemma 2.11, we obtain $\tilde{a}^i \in \tilde{E}_0^{1d}$, $i = 1, 2$. We also have $\tilde{\tau} \asymp \tilde{a}^1$, and $\tilde{\tau} \asymp \tilde{a}^2$. Consequently the weak equivalence $\tilde{a}^1 \asymp \tilde{a}^2$ holds. Applying Lemma 2.3 we can find $N_0 \in \mathbb{N}$ such that $a_n^{(1)} \leq a_n^{(2)}$ and $a_n^{(2)} \leq a_n^{(1)}$ for $n \geq N_0$. Consequently $a_n^{(1)} = a_n^{(2)}$ for $n \geq N_0$ which implies (2.15) for such n . \square

Define the set $\tilde{I}_E^d \subseteq \tilde{I}_E$ by the rule

$$(\{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E^d) \Leftrightarrow (\{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E \text{ and } \{a_n\}_{n \in \mathbb{N}} \text{ is almost decreasing}).$$

Remark 2.13. Let $E \subseteq \mathbb{R}^+$. If $\{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E^d$, then there are $\tilde{\tau} = \{\tau_n\}_{n \in \mathbb{N}} \in \tilde{E}_0^d$ and $\tilde{\beta} = \{\beta_n\}_{n \in \mathbb{N}} \in \tilde{E}_0^d$ such that

$$\lim_{n \rightarrow \infty} \frac{\tau_n}{a_n} = \lim_{n \rightarrow \infty} \frac{\beta_n}{b_n} = 1. \quad (2.16)$$

Definition 2.14. Let $\tilde{A} := \{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E^d$ and $\tilde{L} := \{(l_n, m_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E^d$. We write $\tilde{A} \preceq \tilde{L}$ if there are a natural number $N_1 = N_1(\tilde{A}, \tilde{L})$ and a function $f : \mathbb{N}_{N_1} \rightarrow \mathbb{N}$, where $\mathbb{N}_{N_1} := \{N_1, N_1 + 1, \dots\}$, such that

$$a_n = l_{f(n)} \quad (2.17)$$

for every $n \in \mathbb{N}_{N_1}$. We say that $\tilde{L} \in \tilde{I}_E^d$ is universal if $\tilde{A} \preceq \tilde{L}$ for every $\tilde{A} \in \tilde{I}_E^d$.

The first part of Definition 2.14 can be reformulated as the following.

Proposition 2.15. Let $\tilde{A} = \{(a_n, b_n)\}_{n \in \mathbb{N}}$ and $\tilde{L} = \{(l_n, m_n)\}_{n \in \mathbb{N}}$ belong to \tilde{I}_E^d . $\tilde{A} \preceq \tilde{L}$ if and only if there are $N_1 = N_1(\tilde{A}, \tilde{L})$ and $f : \mathbb{N}_{N_1} \rightarrow \mathbb{N}$ such that

$$b_n = m_{f(n)} \text{ for } n \in \mathbb{N}_{N_1}.$$

Remark 2.16. The universality of $\tilde{L} \in \tilde{I}_E^d$ can be expressed in the language of arrows. Let us denote by Com the set of the connected components of $ExtE$. An element $\tilde{L} \in \tilde{I}_E^d$ is universal if for every $\tilde{A} \in \tilde{I}_E^d$ there are $N_1 \in \mathbb{N}$ and $f : \mathbb{N}_{N_1} \rightarrow \mathbb{N}$ such the diagram

$$\begin{array}{ccccc} \mathbb{N}_{N_1} & \xrightarrow{in} & \mathbb{N} & \xrightarrow{\tilde{A}} & Com \\ & \searrow f & & \nearrow \tilde{L} & \\ & & \mathbb{N} & & \end{array}$$

is commutative. Here in is the natural inclusion of \mathbb{N}_{N_1} in \mathbb{N} , $in(n) = n$ for $n \in \mathbb{N}_{N_1}$.

The following proposition describes the universal elements as the largest elements of the suitable posets.

Proposition 2.17. *Let $E \subseteq \mathbb{R}^+$ be strongly porous on the right at 0 and let $0 \in acE$. The relation \preceq is a preorder on the set \tilde{I}_E^d .*

Proof. We must show that \preceq is reflexive and transitive. The reflexivity of \preceq is evident. To prove that \preceq is transitive note that if $\tilde{A} \preceq \tilde{L}$, then there is an *increasing* function $f : \mathbb{N}_{N_1} \rightarrow \mathbb{N}$ such that (2.17) holds. (The existence of an increasing f meeting (2.17) follows because the sequences $\{a_n\}_{n \in \mathbb{N}}$ and $\{l_n\}_{n \in \mathbb{N}}$ are almost decreasing.) Suppose that $\tilde{A} \preceq \tilde{L}$ and $\tilde{L} \preceq \tilde{T}$, $\tilde{T} = \{(t_n, p_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E^d$. Let $f : \mathbb{N}_{N_1} \rightarrow \mathbb{N}$ and $g : \mathbb{N}_{N_2} \rightarrow \mathbb{N}$ be two functions such that

$$a_n = l_{f(n)} \quad \text{for } n \geq N_1 \quad \text{and} \quad l_n = t_{g(n)} \quad \text{for } n \geq N_2.$$

Put $M := \max\{n \in \mathbb{N} : f(n) \leq N_2\}$. Since f is increasing and unbounded, we have $M < \infty$. Define

$$N_3 := \max\{M, N_1\}$$

with $N_3 := N_1$ if $\{n \in \mathbb{N} : f(n) \leq N_2\} = \emptyset$. Then the inequality $N_3 < \infty$ holds. In accordance with the construction, we have $f(n) \geq N_2$ for every $n \in \mathbb{N}_{N_3}$. Consequently we obtain

$$a_n = l_{f(n)} = t_{g(f(n))}$$

for such n . Thus $\tilde{A} \preceq \tilde{L}$ and $\tilde{L} \preceq \tilde{T}$ imply $\tilde{A} \preceq \tilde{T}$. \square

Using the standard facts from the ordered sets theory we may prove that the pre-order \preceq generates an equivalence \equiv on \tilde{I}_E^d if we put

$$(\tilde{A} \equiv \tilde{T}) \Leftrightarrow (\tilde{A} \preceq \tilde{T} \text{ and } \tilde{T} \preceq \tilde{A}). \quad (2.18)$$

Moreover, if the relations $\tilde{A} \equiv \tilde{S}$ and $\tilde{L} \equiv \tilde{T}$ hold, then $\tilde{A} \preceq \tilde{L}$ if and only if $\tilde{S} \preceq \tilde{T}$. Going over to the factor set induced by \equiv we obtain a partially ordered set (poset). *The preordered set (\tilde{I}_E^d, \preceq) has an universal element if and only if this poset has the largest element.*

Let $\tilde{L} = \{(l_n, m_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E^d$ be universal. Let us define the quantity

$$M = M(\tilde{L}) := \limsup_{n \rightarrow \infty} \frac{l_n}{m_{n+1}}. \quad (2.19)$$

We say that a sequence $\tilde{a} = \{a_n\}_{n \in \mathbb{N}}$, $a_n \in \mathbb{R}$, is *almost strictly decreasing* if $a_{n+1} < a_n$ for sufficiently large n . Write \tilde{I}_E^{sd} for the set of $\{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E^d$ having almost strictly decreasing $\{a_n\}_{n \in \mathbb{N}}$.

Theorem 2.18. *Let $E \subseteq \mathbb{R}^+$ be strongly porous on the right at 0 and let $0 \in acE$. The following conditions are equivalent.*

- (i) E is a **CSP** - set.
- (ii) The preordered set (\tilde{I}_E^d, \preceq) contains an universal element $\tilde{L} = \{(l_n, m_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E^{sd}$ with

$$M(\tilde{L}) < \infty. \quad (2.20)$$

- (iii) E is uniformly strongly porous.

To prove Theorem 2.18 we need some additional lemmas.

Lemma 2.19. *Let $E \subseteq \mathbb{R}^+$. If $\tilde{L} = \{(l_n, m_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E^d$ is universal, then there is a subsequence $\tilde{L}' = \{(l_{n_k}, m_{n_k})\}_{k \in \mathbb{N}}$ of \tilde{L} such that \tilde{L}' is also universal and $\tilde{L}' \in \tilde{I}_E^{sd}$.*

Proof. We construct \tilde{L}' by induction. Since $\{l_n\}_{n \in \mathbb{N}}$ is almost decreasing, there exists $n_1 \in \mathbb{N}$ such that $l_{n+1} \leq l_n$ for $n \geq n_1$. The limit relation $\lim_{n \rightarrow \infty} l_n = 0$ implies that there is $n \geq n_1$ such that $l_n < l_{n_1}$. Write

$$n_2 := \min\{n \in \mathbb{N}_{n_1} : l_n < l_{n_1}\}.$$

Similarly we set

$$n_{k+1} := \min\{n \in \mathbb{N}_{n_k} : l_n < l_{n_k}\} \quad (2.21)$$

for $k = 2, 3, 4, \dots$. For every $n \geq n_1$ there is the unique $k \in \mathbb{N}$ such that

$$n_k \leq n < n_{k+1}. \quad (2.22)$$

Furthermore, the decrease of the sequence $\{l_n\}_{n \in \mathbb{N}_{n_1}}$ implies that

$$l_{n_k} = l_n \quad (2.23)$$

if n satisfies (2.22). Let us define $g : \mathbb{N}_{n_1} \rightarrow \mathbb{N}$ by the rule $g(n) = k$ where k is the unique index satisfying (2.22). In fact, it was proved above that $\tilde{L} \preceq \tilde{L}'$. By Proposition 2.17 the relation \preceq is transitive. Since \tilde{L} is universal, we have $\tilde{T} \preceq \tilde{L}$ for every $\tilde{T} \in \tilde{I}_E^d$. Consequently $\tilde{T} \preceq \tilde{L}'$ for every $\tilde{T} \in \tilde{I}_E^d$, i.e., \tilde{L}' is universal. It still remains to note that (2.21) implies the inequality $l_{n_k} > l_{n_{k+1}}$ for every $k \in \mathbb{N}$. Hence $\{l_{n_k}\}_{k \in \mathbb{N}}$ is a strictly decreasing sequence. Thus $\tilde{L}' \in \tilde{I}_E^{sd}$. \square

Remark 2.20. If $\tilde{L} = \{(l_n, m_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E^{sd}$ and $\tilde{A} = \{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E^{sd}$, then $\tilde{L} \equiv \tilde{A}$ if and only if there exist $N_1, N_2 \in \mathbb{N}$ such that

$$(l_{n+N_1}, m_{n+N_1}) = (a_{n+N_2}, b_{n+N_2})$$

for every $n \in \mathbb{N}$, where \equiv is defined by (2.18).

We do not use this affirmation in the sequel and omit the proof here.

Lemma 2.21. Let E be a **CSP** - set. If $\tilde{L} = \{(l_n, m_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E^{sd}$ is universal, then

$$M(\tilde{L}) = C(E) \quad (2.24)$$

where the quantities $M(\tilde{L})$ and $C(E)$ are defined by (2.19) and (2.11) respectively.

Proof. Let $\tilde{L} \in \tilde{I}_E^{sd}$ be universal. We shall first prove the inequality

$$M(\tilde{L}) \geq C(E). \quad (2.25)$$

Inequality (2.25) holds if and only if

$$M(\tilde{L}) \geq C(\tilde{\tau}) \quad (2.26)$$

for every $\tilde{\tau} \in \tilde{E}_0^d$, where $C(\tilde{\tau})$ was defined in (2.11). Let $\tilde{\tau} \in \tilde{E}_0^d$. By condition of the lemma, E is completely strongly porous. Hence there is $\{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E$ such that $\tilde{\tau} \asymp \tilde{a}$. By Lemma 2.3 we have the inequality

$$\limsup_{n \rightarrow \infty} \frac{a_n}{\tau_n} < \infty \quad (2.27)$$

and, for sufficiently large n , the inequality

$$\tau_n \leq a_n. \quad (2.28)$$

Proposition 2.12 and the definition of $C(\tilde{\tau})$ imply

$$C(\tilde{\tau}) = \limsup_{n \rightarrow \infty} \frac{a_n}{\tau_n}. \quad (2.29)$$

Hence to prove (2.26) we must show that

$$M(\tilde{L}) \geq \limsup_{n \rightarrow \infty} \frac{a_n}{\tau_n}. \quad (2.30)$$

Be Lemma 2.11 we have

$$\tilde{A} := \{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E^d. \quad (2.31)$$

Since \tilde{L} is universal, from (2.31) follows that $\tilde{A} \preceq \tilde{L}$. Consequently there are $N_1 \in \mathbb{N}$ and the increasing function $f : \mathbb{N}_{N_1} \rightarrow \mathbb{N}$ such that

$$a_n \geq a_{n+1} \quad \text{and} \quad a_n = l_{f(n)} \quad (2.32)$$

for $n \geq N_1$. Since $\tilde{L} = \{(l_n, m_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E^{sd}$, we may suppose that $\tilde{l} = \{l_n\}_{n \in \mathbb{N}}$ is strictly decreasing. Replacing $\tilde{\tau}$ by a suitable subsequence we may assume that $\tilde{\tau}$ and \tilde{a} are also strictly decreasing, f is strictly increasing, and that the relations

$$\tau_1 \leq l_1, \quad \lim_{n \rightarrow \infty} \frac{a_n}{\tau_n} = \limsup_{n \rightarrow \infty} \frac{a_n}{\tau_n} \quad (2.33)$$

hold. The closed intervals $[m_{n+1}, l_n]$, $n = 1, 2, \dots$, together with the half-open interval $[m_1, \infty)$ form a cover of the set $E_0 = E \setminus \{0\}$, i.e.

$$E_0 \subseteq [m_1, \infty) \cup \left(\bigcup_{n \in \mathbb{N}} [m_{n+1}, l_n] \right).$$

Since the elements of this cover are pairwise disjoint and $\tau_1 \leq l_1$, for every $n \in \mathbb{N}$ there is a unique $k(n) \in \mathbb{N}$ such that

$$\tau_n \in [m_{k(n)+1}, l_{k(n)}]. \quad (2.34)$$

We claim that the equality

$$k(n) = f(n) \quad (2.35)$$

holds for sufficiently large n . Indeed, using (2.28), (2.32) and (2.34) we obtain

$$\tau_n \leq l_{f(n)} \quad \text{and} \quad \tau_n \geq m_{k(n)+1}.$$

These inequalities and

$$m_{k(n)+1} > l_{k(n)+1} > l_{k(n)+2} > l_{k(n)+3} > \dots$$

imply

$$f(n) \leq k(n). \quad (2.36)$$

Suppose that the last inequality is strict for n belonging to an infinite set $A \subseteq \mathbb{N}$, i.e.

$$f(n) \leq k(n) - 1 \quad (2.37)$$

for $n \in A$. Since $\{a_n\}_{n \in \mathbb{N}} \asymp \{\tau_n\}_{n \in \mathbb{N}}$ and $a_n = l_{f(n)}$, we can find a constant $c \in (0, 1)$ such that

$$cl_{f(n)} \leq \tau_n \leq l_{f(n)} \quad (2.38)$$

for sufficiently large n . From (2.34), (2.36) and (2.38) it follows that

$$cl_{f(n)} \leq \tau_n \leq l_{k(n)} \leq l_{f(n)}. \quad (2.39)$$

Since $\tilde{l} = \{l_n\}_{n \in \mathbb{N}}$ is strictly increasing and $(l_n, m_n) \cap (l_j, m_j) = \emptyset$ if $n \neq j$, (2.37) implies that

$$l_{k(n)} < m_{k(n)} \leq l_{k(n)-1} \leq l_{f(n)} < m_{f(n)}.$$

These inequalities and (2.39) show that

$$cl_{f(n)} \leq \tau_n \leq l_{k(n)} < m_{k(n)} \leq l_{k(n)-1} < l_{f(n)}$$

for $n \in A$. Consequently we have

$$\frac{1}{c} = \lim_{n \rightarrow \infty} \frac{l_{f(n)}}{cl_{f(n)}} \geq \limsup_{n \rightarrow \infty, n \in A} \frac{m_{k(n)}}{l_{k(n)}},$$

contrary to the limit relation

$$\lim_{n \rightarrow \infty} \frac{m_n}{l_n} = \infty.$$

Hence the set of $n \in \mathbb{N}$ meeting the condition $f(n) < k(n)$ is finite. Thus (2.35) holds for sufficiently large n .

Now it is easy to prove (2.30). By (2.32) and (2.35) we have

$$a_n = l_{f(n)} = l_{k(n)}.$$

Relation (2.34) implies $\tau_n \geq m_{k(n)+1}$. Consequently

$$\frac{a_n}{\tau_n} \leq \frac{l_{k(n)}}{m_{k(n)+1}}.$$

Hence

$$\limsup_{n \rightarrow \infty} \frac{a_n}{\tau_n} \leq \limsup_{n \rightarrow \infty} \frac{l_{k(n)}}{m_{k(n)+1}} \leq \limsup_{n \rightarrow \infty} \frac{l_n}{m_{n+1}} = M(\tilde{L}).$$

Inequality (2.30) follows, so that (2.25) is proved.

To prove the inequality

$$M(\tilde{L}) \leq C(E) \tag{2.40}$$

we take a sequence $\tilde{\tau} = \{\tau_n\}_{n \in \mathbb{N}} \in \tilde{E}_0^d$ such that (2.34) holds with $k(n) = n$ and

$$\lim_{n \rightarrow \infty} \frac{m_{n+1}}{\tau_n} = 1. \tag{2.41}$$

A desirable $\tilde{\tau}$ can be constructed as in the proof of Proposition 2.6. The set E is $\tilde{\tau}$ -strongly porous because E is a **CSP** - set. Hence there is $\tilde{a} \asymp \tilde{\tau}$ such that $\{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E^d$. By Lemma 2.11 the sequence \tilde{a} is almost decreasing. Since $\tau_n \in [m_{n+1}, l_n]$, using (2.35) we obtain

$$a_n = l_n$$

for sufficiently large n . From (2.29) and (2.41) it follows that

$$\begin{aligned} C(\tilde{\tau}) &= \limsup_{n \rightarrow \infty} \frac{a_n}{\tau_n} = \limsup_{n \rightarrow \infty} \frac{l_n}{m_{n+1}} \frac{m_{n+1}}{\tau_n} = \limsup_{n \rightarrow \infty} \frac{l_n}{m_{n+1}} \lim_{n \rightarrow \infty} \frac{m_{n+1}}{\tau_n} \\ &= \limsup_{n \rightarrow \infty} \frac{l_n}{m_{n+1}} = M(\tilde{L}). \end{aligned} \tag{2.42}$$

Since $C(E) \geq C(\tilde{\tau})$, inequality (2.40) follows.

To complete the proof, it suffices to observe that (2.25) and (2.40) imply (2.24). \square

Directly from (2.42) we obtain

Corollary 2.22. Let $E \subseteq \mathbb{R}^+$ be a **CSP** - set. If $\tilde{L} = \{(l_n, m_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E^{sd}$ is universal, then $M(\tilde{L}) < \infty$.

Remark 2.23. As has been shown in Lemma 2.21 the equality $M(\tilde{L}) = C(E)$ holds for every universal $\tilde{L} \in \tilde{I}_E^{sd}$. Suppose that $\tilde{L} \in \tilde{I}_E^d$ is universal but $\tilde{L} \notin \tilde{I}_E^{sd}$. Define the set $A \subseteq \mathbb{N}$ by the rule

$$(n \in A) \Leftrightarrow (n \in \mathbb{N} \text{ and } (l_{n+1}, m_{n+1}) = (l_n, m_n)).$$

Let $\tilde{L}' \in \tilde{I}_E^{sd}$ be the universal element of (\tilde{I}_E^d, \preceq) constructed from \tilde{L} as in Lemma 2.19. Using the definition of the set A we obtain

$$\begin{aligned} M(\tilde{L}) &= \limsup_{n \rightarrow \infty} \frac{l_{n+1}}{m_n} = \limsup_{n \rightarrow \infty, n \in A} \frac{l_{n+1}}{m_n} \vee \limsup_{n \rightarrow \infty, n \in \mathbb{N} \setminus A} \frac{l_{n+1}}{m_n} \\ &= \limsup_{n \rightarrow \infty, n \in A} \frac{l_n}{m_n} \vee M(\tilde{L}') = 0 \vee M(\tilde{L}') = M(\tilde{L}'). \end{aligned}$$

Consequently if $\tilde{L}, \tilde{S} \in \tilde{I}_E^d$ are universal, then $M(\tilde{L}) = M(\tilde{S})$. Thus condition (ii) of Theorem 2.18 can be formulated by the following equivalent way.

- The set of universal elements $\tilde{L} \in \tilde{I}_E^d$ is nonempty and the inequality $M(\tilde{L}) < \infty$ holds for every \tilde{L} from this set.

Proof of Theorem 2.18. (i) \Rightarrow (ii). Let E be a **CSP** - set. We shall first prove that there is a sequence $\tilde{u} = \{u_n\}_{n \in \mathbb{N}} \in \tilde{E}_0^d$ such that for every $\tilde{\tau} = \{\tau_k\}_{k \in \mathbb{N}} \in \tilde{E}_0^d$ can be found an almost increasing function $f : \mathbb{N} \rightarrow \mathbb{N}$ satisfying the relation

$$\{\tau_k\}_{k \in \mathbb{N}} \asymp \{u_{f(k)}\}_{k \in \mathbb{N}}. \quad (2.43)$$

Let us define the sequence of sets E_j , $j \in \mathbb{N}$, by the rule

$$E_1 := E \cap [1, \infty), E_2 := E \cap [\frac{1}{2}, 1), \dots, E_j := E \cap [\frac{1}{2^{j-1}}, \frac{1}{2^{j-2}}). \quad (2.44)$$

There is the unique subsequence $\{E_{j_n}\}_{n \in \mathbb{N}}$ of the sequence $\{E_j\}_{j \in \mathbb{N}}$ such that

$$E \setminus \{0\} = \bigcup_{n \in \mathbb{N}} E_{j_n} \quad \text{and} \quad E_{j_n} \neq \emptyset$$

for every $n \in \mathbb{N}$. For convenience we set $A_n := E_{j_n}$, $n \in \mathbb{N}$. Let $\{u_n\}_{n \in \mathbb{N}}$ be a sequence of positive real numbers meeting the condition $u_n \in A_n$ for every $n \in \mathbb{N}$. It is clear that $\{u_n\}_{n \in \mathbb{N}} \in \tilde{E}_0^d$. For every $\tilde{\tau} = \{\tau_k\}_{k \in \mathbb{N}} \in \tilde{E}_0^d$, define $f : \mathbb{N} \rightarrow \mathbb{N}$ by the rule

$$f(k) = n \quad \text{if and only if} \quad \tau_k \in A_n.$$

The function f is well-defined because

$$E \setminus \{0\} = \bigcup_{n \in \mathbb{N}} A_n \quad \text{and} \quad A_j \cap A_i = \emptyset \quad \text{if} \quad i \neq j.$$

It follows directly from (2.44) that

$$\frac{1}{2} \tau_k \leq u_{f(k)} \leq 2 \tau_k$$

if $f(k) \geq 2$. Moreover, since $\tilde{\tau}$ and \tilde{u} are almost decreasing and $\lim_{n \in \mathbb{N}} \tau_n = 0$, the function $f : \mathbb{N} \rightarrow \mathbb{N}$ is almost increasing and the set $\{k \in \mathbb{N} : f(k) = 1\}$ is finite. Consequently there are some constants c_1, c_2 such that

$$c_2 \tau_k \leq u_{f(k)} \leq c_1 \tau_k$$

for all $k \in \mathbb{N}$. Thus (2.43) holds.

Let $\{u_n\}_{n \in \mathbb{N}} \in \tilde{E}_0^d$ be the sequence constructed above. Since E is a **CSP** - set, E is \tilde{u} -strongly porous. Hence, there is $\tilde{A} := \{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E$ such that

$$\tilde{a} \asymp \tilde{u}. \quad (2.45)$$

Lemma 2.11 implies that \tilde{a} is almost decreasing, i.e., $\tilde{A} \in \tilde{I}_E^d$. We claim that \tilde{A} is universal. Indeed, as was shown for every $\tilde{\tau} = \{\tau_k\}_{k \in \mathbb{N}} \in \tilde{E}_0^d$ there is $f : \mathbb{N} \rightarrow \mathbb{N}$ such that (2.43) holds. The relation $\{u_n\}_{n \in \mathbb{N}} \asymp \{a_n\}_{n \in \mathbb{N}}$ implies that

$$\{u_{f(k)}\}_{k \in \mathbb{N}} \asymp \{a_{f(k)}\}_{k \in \mathbb{N}}. \quad (2.46)$$

Every interval $(a_{f(n)}, b_{f(n)})$ is a connected component of $Ext E$ and, in addition, $\lim_{n \rightarrow \infty} \frac{b_n}{a_n} = \infty$ implies $\lim_{k \rightarrow \infty} \frac{b_{f(k)}}{a_{f(k)}} = \infty$ because $\lim_{n \rightarrow \infty} f(n) = \infty$. Consequently we obtain

$$\{(a_{f(k)}, b_{f(k)})\}_{k \in \mathbb{N}} \in \tilde{I}_E. \quad (2.47)$$

Moreover, since f is almost increasing and $\tilde{A} = \{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E^d$, (2.47) implies

$$\{(a_{f(k)}, b_{f(k)})\}_{k \in \mathbb{N}} \in \tilde{I}_E^d. \quad (2.48)$$

From (2.43) and (2.46) we obtain

$$\{\tau_k\}_{k \in \mathbb{N}} \asymp \{a_{f(k)}\}_{k \in \mathbb{N}}. \quad (2.49)$$

Using (2.48), (2.49) and Remark 2.13, we can prove that $\tilde{L} \preceq \tilde{A}$ for every $\tilde{L} \in \tilde{I}_E^d$, as required.

By Lemma 2.19 we can find an universal element $\tilde{L} \in \tilde{I}_E^{sd}$. In according with Corollary 2.22 we have $M(\tilde{L}) < \infty$. Thus condition (i) implies (ii).

The implication (iii) \Rightarrow (i) is evident. Moreover, using Lemma 2.21, we can simply verify that the implication ((i)&(ii)) \Rightarrow (iii) is true. Consequently to complete the proof it suffices to show that (ii) \Rightarrow (i). Suppose that condition (ii) holds. Let $\tilde{\tau} = \{\tau_n\}_{n \in \mathbb{N}} \in \tilde{E}_0^d$ and let $\tilde{L} = \{(l_k, m_k)\}_{k \in \mathbb{N}} \in \tilde{I}_E^{sd}$ be universal. As in the proof of Lemma 2.21 we may suppose that $\{l_n\}_{n \in \mathbb{N}}$ is a strictly decreasing sequence and that $\tau_1 \leq l_1$. Then for every $n \in \mathbb{N}$ there is a unique $k(n) \in \mathbb{N}$ such that

$$m_{k(n)+1} \leq \tau_n \leq l_{k(n)}, \quad (2.50)$$

(see (2.34)). Double inequality (2.50) implies

$$\limsup_{n \rightarrow \infty} \frac{l_{k(n)}}{\tau_n} \leq \limsup_{n \rightarrow \infty} \frac{l_{k(n)}}{m_{k(n)+1}} \leq \limsup_{k \rightarrow \infty} \frac{l_k}{m_{k+1}} = M(\tilde{L}) < \infty.$$

Since $\{(l_{k(n)}, m_{k(n)})\}_{n \in \mathbb{N}} \in \tilde{I}_E^d$, the set E is $\tilde{\tau}$ -strongly porous by Lemma 2.3. Thus condition (i) follows from condition (ii). \square

Remark 2.24. Conditions (i) and (iii) from Theorem 2.18 are equivalent for arbitrary $E \subseteq \mathbb{R}^+$. Indeed, if $p^+(E, 0) < 1$, then both (i) and (iii) are evidently false. If $p^+(E, 0) = 1$ but $0 \notin acE$, then (i) and (iii) are true (see Remark 1.5 and Remark 2.9). In this connection it should be pointed out that condition (ii) of Theorem 2.18 implies $\tilde{I}_E \neq \emptyset$. Consequently, if (ii) holds, then $0 \in acE$ and $p^+(E, 0) = 1$ (see Remark 1.3).

The following example shows that the existence of an universal $\tilde{L} \in \tilde{I}_E^{sd}$ does not imply the inequality $M(\tilde{L}) < \infty$.

Example 2.25. Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence of strictly decreasing positive real numbers such that

$$\lim_{n \rightarrow \infty} \frac{x_{n+1}}{x_n} = 0.$$

Define the set E as $\bigcup_{n \in \mathbb{N}} [x_{2n+1}, x_{2n}]$. It follows from Lemma 2.3 that $E \notin \mathbf{CSP}$ but, as easily seen, the sequence $\{(x_{2n}, x_{2n-1})\}_{n \in \mathbb{N}} \in \tilde{I}_E^{sd}$ is universal. (See Fig. 1.).

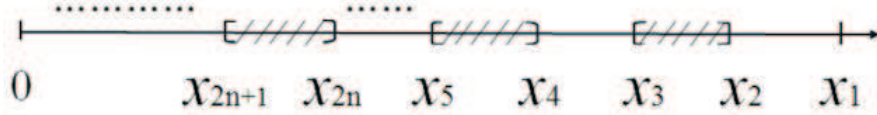


Fig. 1. The set E is shaded here

The next theorem describes the structure of sets $E \subseteq \mathbb{R}^+$ for which there is an universal $\tilde{L} \in \tilde{I}_E^{sd}$.

As in Remark 2.16 write Com for the set of all connected components of $ExtE$.

Theorem 2.26. Let $E \subseteq \mathbb{R}^+$ be strongly porous on the right at 0 and let $0 \in acE$. The preordered set (\tilde{I}_E^d, \preceq) contains an universal element if and only if there is a constant $c > 1$ such that for every $K > 1$ there is $t > 0$ for which the inequalities $t > a$ and $\frac{b}{a} > c$ imply the inequality $\frac{b}{a} > K$ for every $(a, b) \in Com$.

Proof. Suppose that there is an universal element $\tilde{L} = \{(l_n, m_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E^d$. We must prove that

$$\exists c > 1 \forall K > 1 \exists t > 0 \forall (a, b) \in Com : (a < t) \& \left(\frac{b}{a} > c \right) \Rightarrow \left(\frac{b}{a} > K \right). \quad (2.51)$$

By Lemma 2.19 we may assume that $\{l_n\}_{n \in \mathbb{N}}$ is strictly decreasing. Using the limit relations

$$\lim_{n \rightarrow \infty} \frac{m_n}{l_n} = \infty \text{ and } \lim_{n \rightarrow \infty} l_n = 0$$

and the strict decrease of $\{l_n\}_{n \in \mathbb{N}}$ we obtain that

$$\forall K > 1 \exists t > 0 : \forall n \in \mathbb{N} (l_n < t) \Rightarrow \left(\frac{m_n}{l_n} > K \right). \quad (2.52)$$

If (2.51) does not hold, then

$$\forall c > 1 \exists K = K(c) > 1 \forall t > 0 \exists (a, b) \in Com : (t > a) \& \left(c < \frac{b}{a} \leq K(c) \right). \quad (2.53)$$

Using this formula with $c = j$ and $K = K(j)$, for $j = 1, 2, \dots$, we see that

$$\forall t > 0 \exists (a_j, b_j) \in Com : (a_j < t) \& \left(j \leq \frac{b_j}{a_j} \leq K(j) \right). \quad (2.54)$$

Formula (2.52) implies that

$$\forall n \in \mathbb{N} \exists t_j > 0 : (l_n < t_j) \Rightarrow \left(\frac{m_n}{l_n} > K(j) \right). \quad (2.55)$$

We can suppose also that $\lim_{j \rightarrow \infty} t_j = 0$ and $\{t_j\}_{j \in \mathbb{N}}$ is strictly decreasing. From (2.54) with $t = t_j$ it follows that

$$\forall j \in \mathbb{N} \exists (a_j, b_j) \in Com : (a_j < t_j) \ \& \ \left(j \leq \frac{b_j}{a_j} \leq K(j) \right). \quad (2.56)$$

Consequently the sequence $\tilde{A} := \{(a_j, b_j)\}_{j \in \mathbb{N}}$ belongs to \tilde{I}_E . Using the limit relation $\lim_{j \rightarrow \infty} t_j = 0$ and passing on to a suitable subsequence we can also claim that $\tilde{A} \in \tilde{I}_E^d$. Formulas (2.55) and (2.56) imply that

$$(a_j, b_j) \neq (l_n, m_n)$$

for every element (l_n, m_n) of \tilde{L} . Consequently \tilde{L} is not universal, contrary to the assumption.

Conversly, suppose that (2.51) holds. Let us prove that there exists an universal element in \tilde{I}_E^d . Let c be the constant satisfying (2.51). Define a subset $Com(c)$ of the set Com by the rule

$$((a, b) \in Com(c)) \Leftrightarrow \left((a, b) \in Com, a > 1 \text{ and } \frac{b}{a} > c \right).$$

We can enumerate of the intervals $(a, b) \in Com(c)$ in the sequence

$$(a_1, b_1), (a_2, b_2), \dots, (a_n, b_n), \dots$$

such that $\{a_n\}_{n \in \mathbb{N}}$ is strictly decreasing. Condition (2.51) implies that $\tilde{A} = \{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E^d$. The universality of \tilde{A} follows directly from Definition 2.14 and 2.51. \square

As in Remark 2.24 it should be noted that the existence of an universal $\tilde{L} \in \tilde{I}_E^d$ implies that $0 \in acE$ and $p^+(E, 0) = 1$.

An illustrating model to Theorem 2.26. Let $E \subseteq (0, 1]$ be closed and let $0 \in acE$. Write

$$W = -\ln E := \left\{ \ln \left(\frac{1}{x} \right) : x \in E \right\}.$$

We can consider W as “a photography of an one-dimensioned liquid” with some “gas bubbles” $(\ln(\frac{1}{b}), \ln(\frac{1}{a}))$, where $(a, b) \in Com$, which move to $+\infty$. Theorem 2.26 means that there is a critical value $\ln c$ such that if the size of gas bubbles are greater than $\ln c$, then these bubbles undergo an unbounded blow up during theirs motion.

The following simple proposition can be considered as a limit case of Theorem 2.26.

Proposition 2.27. *Let $E \subseteq \mathbb{R}^+$ and $\tilde{L} = \{(l_n, m_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E^d$. Suppose that for every $(a, b) \in Com$ there is $n \in \mathbb{N}$ such that $(a, b) = (l_n, m_n)$. Then \tilde{L} is universal.*

The proof follows directly from Definition 2.14.

3 Another characterizations of CSP - sets

Let E be a subset of \mathbb{R}^+ . Define the set $\tilde{H} = \tilde{H}(E)$ of the sequences $\tilde{h} = \{h_n\}_{n \in \mathbb{N}}$, $h_n > 0$, $\lim_{n \rightarrow \infty} h_n = 0$ by the rule:

$$(\tilde{h} \in \tilde{H}) \Leftrightarrow \left(\frac{\lambda(E, 0, h_n)}{h_n} \rightarrow p^+(E, 0) \text{ with } n \rightarrow \infty \right) \quad (3.1)$$

where the quantities $p^+(E, 0)$ and $\lambda(E, 0, h_n)$ are the same as in Definition 1.1.

Theorem 3.1. *Let $E \subseteq \mathbb{R}^+$ be strongly porous on the right at 0. Then E is a **CSP** - set if and only if for every $\tilde{\tau} = \{\tau_n\}_{n \in \mathbb{N}} \in \tilde{E}_0^d$ there is $\tilde{h} = \{h_n\}_{n \in \mathbb{N}} \in \tilde{H}(E)$ such that $\tilde{\tau} \asymp \tilde{h}$.*

Proof. The necessity is easy to prove. Suppose E is a **CSP** - set. Let $\tilde{\tau} \in \tilde{E}_0^d$. By Theorem 2.18 there is an universal element $\tilde{L} = \{(l_n, m_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E^{sd}$ with $M(\tilde{L}) < \infty$. Reasoning as in the proof of Theorem 2.18, we can find $k(n) \in \mathbb{N}$ such that

$$\tau_n \in [m_{k(n)+1}, l_{k(n)}] \quad (3.2)$$

for sufficiently large n (see (2.34)). Membership (3.2) implies the inequalities

$$m_{k(n)+1} \leq \tau_n \quad \text{and} \quad \frac{\tau_n}{m_{k(n)+1}} \leq \frac{l_{k(n)}}{m_{k(n)+1}}.$$

Thus we have

$$\limsup_{n \rightarrow \infty} \frac{\tau_n}{m_{k(n)+1}} \leq \limsup_{n \rightarrow \infty} \frac{l_{k(n)}}{m_{k(n)+1}} \leq M(\tilde{L}) < \infty.$$

Consequently there are $c_1 \geq 1$ and $N_1 \in \mathbb{N}$ such that $m_{k(n)+1} \leq \tau_n \leq c_1 m_{k(n)+1}$ for $n \geq N_1$. If we set $m_{k(n)+1} := m_{k(N_1)+1}$ for $n < N_1$, then it is easy to see that $\{\tau_n\}_{n \in \mathbb{N}} \asymp \{m_{k(n)+1}\}_{n \in \mathbb{N}}$. To be certain that $\{m_{k(n)+1}\}_{n \in \mathbb{N}} \in \tilde{H}(E)$, it suffices to check that

$$\lim_{n \rightarrow \infty} \frac{\lambda(E, 0, m_{k(n)+1})}{m_{k(n)+1}} = 1. \quad (3.3)$$

(Indeed, $p^+(E, 0) = 1$ because E is strongly porous on the right at 0.) Since the quantity $\lambda(E, 0, m_{k(n)+1})$ is the length of the largest open interval in the set $(0, m_{k(n)+1}) \cap \text{Ext}E$ and

$$(l_{k(n)+1}, m_{k(n)+1}) \subseteq (0, m_{k(n)+1}) \cap \text{Ext}E,$$

we have

$$\frac{m_{k(n)+1} - l_{k(n)+1}}{m_{k(n)+1}} \leq \frac{\lambda(E, 0, m_{k(n)+1})}{m_{k(n)+1}} \leq 1. \quad (3.4)$$

The sequence \tilde{L} belongs to \tilde{I}_E^{sd} . Hence

$$\lim_{n \rightarrow \infty} \frac{m_{k(n)+1} - l_{k(n)+1}}{m_{k(n)+1}} = 1.$$

The last relation and (3.4) imply (3.3).

The proof of the sufficiency is more awkward, so we divide it into several lemmas.

Lemma 3.2. *Let $E \subseteq \mathbb{R}^+$ be strongly porous on the right at 0 and let $\tilde{\tau} = \{\tau_n\}_{n \in \mathbb{N}} \in \tilde{E}_0^d$ and $\tilde{h} = \{h_n\}_{n \in \mathbb{N}} \in \tilde{H}(E)$. If $\tilde{\tau} \asymp \tilde{h}$, then there is $\{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E$ such that*

$$\{\tau_n\}_{n \in \mathbb{N}} \asymp \{b_n\}_{n \in \mathbb{N}}. \quad (3.5)$$

Proof. Let $\tilde{\tau} \asymp \tilde{h}$. By the definition of $\tilde{H}(E)$, for every $n \in \mathbb{N}$, there is an interval $(a'_n, b'_n) \subseteq (0, h_n) \cap \text{Ext}E$ such that

$$\lim_{n \rightarrow \infty} \frac{b'_n - a'_n}{h_n} = 1. \quad (3.6)$$

Moreover, the relation $\tilde{\tau} \asymp \tilde{h}$ implies that there are constants $k \in (0, 1)$ and $K \in (1, \infty)$ such that

$$\tau_n \in (kh_n, Kh_n) \quad (3.7)$$

for every $n \in \mathbb{N}$. Consequently

$$\tau_n \in (0, Kh_n) \setminus (b'_n - a'_n). \quad (3.8)$$

Using (3.6) we can show that

$$b'_n > kh_n > a'_n \quad (3.9)$$

for sufficiently large n . It is clear that $Kh_n > h_n \geq b'_n$. Hence (3.7) – (3.9) imply

$$\tau_n \in [b'_n, Kh_n) \quad (3.10)$$

for sufficiently large n (see Fig. 2 below).

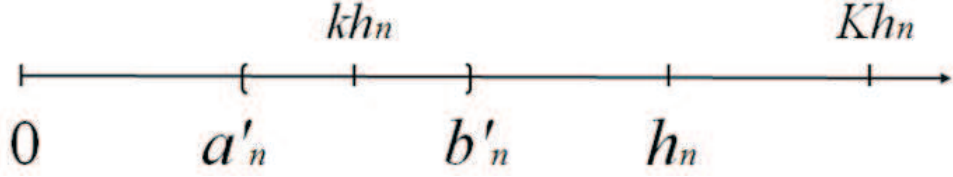


Fig. 2

Let (a_n, b_n) be the connected component of $ExtE$ meeting the inclusion $(a'_n, b'_n) \subseteq (a_n, b_n)$. From (3.10) it follows $\tau_n \geq b_n$. Hence

$$kh_n < b'_n \leq b_n \leq \tau_n < Kh_n \quad (3.11)$$

for sufficiently large n . Consequently $\tilde{\tau} \asymp \tilde{h}$ and $\tilde{b} \asymp \tilde{h}$, so that (3.5) follows. To complete the proof, it suffices to show the membership $\{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E$. The last relation holds if and only if

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0. \quad (3.12)$$

Inequalities $a_n \leq a'_n < b'_n \leq b_n$ imply that

$$0 \leq \frac{a_n}{b_n} \leq \frac{a'_n}{b'_n}. \quad (3.13)$$

Moreover, since

$$\frac{b'_n - a'_n}{h_n} \leq \frac{b'_n - a'_n}{b'_n} \leq 1,$$

limit relation (3.6) yields

$$\lim_{n \rightarrow \infty} \frac{a'_n}{b'_n} = 0.$$

Thus (3.13) follows from (3.12). □

Remark 3.3. It is clear that $\{b_n\}_{n \in \mathbb{N}} \in \tilde{H}(E)$ for $\{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E$.

The following lemmas are analogs of Lemma 2.11, Proposition 2.12 and have the similar proofs.

Lemma 3.4. *Let $E \subseteq \mathbb{R}^+$. If $\tilde{\tau} \in \tilde{E}_0^d$ and $\{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E$, then the weak equivalence $\tilde{b} \asymp \tilde{\tau}$ implies that \tilde{b} and \tilde{a} are almost decreasing.*

Lemma 3.5. Let $E \subseteq \mathbb{R}^+$, $\tilde{\tau} = \{\tau_n\}_{n \in \mathbb{N}} \in \tilde{E}_0^d$, and let $\{(a_n^{(i)}, b_n^{(i)})\}_{n \in \mathbb{N}} \in \tilde{I}_E$, $i = 1, 2$. If

$$\tilde{b}^1 \asymp \tilde{\tau} \asymp \tilde{b}^2$$

where $\tilde{b}^i = \{b_n^{(i)}\}_{n \in \mathbb{N}}$, $i = 1, 2$, then there is $N_0 \in \mathbb{N}$ such that

$$(a_n^{(1)}, b_n^{(1)}) = (a_n^{(2)}, b_n^{(2)})$$

for every $n \geq N_0$.

The next lemma is closely related to the implication (i) \Rightarrow (ii) from Theorem 2.18.

Lemma 3.6. Let $E \subseteq \mathbb{R}^+$ be strongly porous on the right at 0 and let $0 \in acE$. If for every $\tilde{\tau} = \{\tau_n\}_{n \in \mathbb{N}} \in \tilde{E}_0^d$ there is $\{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E$ such that $\{\tau_n\}_{n \in \mathbb{N}} \asymp \{b_n\}_{n \in \mathbb{N}}$, then there is an universal $\tilde{L} = \{(l_n, m_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E^d$ with

$$M(\tilde{L}) < \infty. \quad (3.14)$$

The following proof is a modification of the corresponding part of the proof of Theorem 2.18.

Proof of Lemma 3.6. Suppose that for every $\tilde{\tau} = \{\tau_n\}_{n \in \mathbb{N}} \in \tilde{E}_0^d$ there is $\{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E$ such that $\tilde{\tau} \asymp \tilde{b} = \{b_n\}_{n \in \mathbb{N}}$. In the proof of Theorem 2.18 we have found a sequence $\tilde{u} = \{u_n\}_{n \in \mathbb{N}} \in \tilde{E}_0^d$ such that for every $\tilde{\tau} \in \tilde{E}_0^d$ there is an almost increasing function $f : \mathbb{N} \rightarrow \mathbb{N}$ satisfying the relation

$$\{\tau_k\}_{k \in \mathbb{N}} \asymp \{u_{f(k)}\}_{k \in \mathbb{N}}. \quad (3.15)$$

By the supposition there is $\{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E$ such that

$$\tilde{u} \asymp \tilde{b}. \quad (3.16)$$

Since $\tilde{u} \in \tilde{E}_0^d$, Lemma 3.4 implies that \tilde{b} and \tilde{u} are almost decreasing. Consequently $\tilde{A} := \{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E^d$. We shall show that \tilde{A} is universal. Let $\tilde{L} = \{(l_n, m_n)\}_{n \in \mathbb{N}}$ be an arbitrary element of \tilde{I}_E^d . Using Definition 2.15 we see that \tilde{A} is universal if and only if there are $N_1 \in \mathbb{N}$ and $f : \mathbb{N}_{N_1} \rightarrow \mathbb{N}$ such that

$$m_n = b_{f(n)} \quad (3.17)$$

for $n \in \mathbb{N}_{N_1}$. It is easy to show that there is $\tilde{\tau} = \{\tau_n\}_{n \in \mathbb{N}} \in \tilde{E}_0^d$ such that

$$\lim_{n \rightarrow \infty} \frac{\tau_n}{m_n} = 1. \quad (3.18)$$

The last limit relation implies that $\{m_n\}_{n \in \mathbb{N}} = \tilde{m} \asymp \tilde{\tau} = \{\tau_n\}_{n \in \mathbb{N}}$. This equivalence, (3.15) and (3.16) give us

$$\{m_k\}_{k \in \mathbb{N}} \asymp \{b_{f(k)}\}_{k \in \mathbb{N}}.$$

It is clear that $\{(a_{f(k)}, b_{f(k)})\}_{k \in \mathbb{N}} \in \tilde{I}_E^d$. Consequently, by Lemma 3.5, there is $N_0 \in \mathbb{N}$ such that

$$(l_k, m_k) = (a_{f(k)}, b_{f(k)})$$

for all $k \geq N_0$. Equality (3.17) follows for sufficiently large n . Hence $\tilde{A} \in \tilde{I}_E^d$ is universal. Using Lemma 2.19 we can assume that $\{a_n\}_{n \in \mathbb{N}}$ and $\{b_n\}_{n \in \mathbb{N}}$ are strictly decreasing. To complete the proof it suffices to show that $M(\tilde{A}) < \infty$. As in the proof of Lemma 2.21 we may consider the closed intervals $[b_{n+1}, a_n]$, $n = 1, 2, \dots$, that together with the half-open interval $[b_1, \infty)$ form a disjoint cover of the set $E \setminus \{0\}$,

$$E \setminus \{0\} \subseteq [b_1, \infty) \cup \left(\bigcup_{n \in \mathbb{N}} [b_{n+1}, a_n] \right).$$

We can find a sequence $\tilde{\tau} = \{\tau_n\}_{n \in \mathbb{N}} \in \tilde{E}_0^d$ such that

$$\lim_{n \rightarrow \infty} \frac{\tau_n}{a_n} = 1 \quad \text{and} \quad \tau_n \in [b_{n+1}, a_n] \quad (3.19)$$

for every $n \in \mathbb{N}$. Reasoning as in the proof of equality (2.35) we can see that

$$\{\tau_n\}_{n \in \mathbb{N}} \asymp \{b_{n+1}\}_{n \in \mathbb{N}},$$

i.e., there are positive constants c_1, c_2 such that

$$c_1 b_{n+1} \leq \tau_n \leq c_2 b_{n+1}.$$

The last inequality and (3.19) imply

$$\infty > c_2 \geq \limsup_{n \rightarrow \infty} \frac{\tau_n}{b_{n+1}} = \limsup_{n \rightarrow \infty} \frac{\tau_n}{a_n} \frac{a_n}{b_{n+1}} = \limsup_{n \rightarrow \infty} \frac{a_n}{b_{n+1}} = M(\tilde{A}),$$

and so the lemma is proved. \square

Now we can simply finish the proof of Theorem 3.1.

Proof of Theorem 3.1. The sufficiency. Suppose for every $\tilde{\tau} = \{\tau_n\}_{n \in \mathbb{N}} \in \tilde{E}_0^d$ there is $\tilde{h} = \{h_n\}_{n \in \mathbb{N}} \in \tilde{H}(E)$ such that $\tilde{\tau} \asymp \tilde{h}$. Then, by Lemma 3.2, for every $\tilde{\tau} \in \tilde{E}_0^d$ there is $\{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E$ such that $\tilde{\tau} \asymp \tilde{b}$. Consequently, by Lemma 3.6, the preordered set (\tilde{I}_E^d, \preceq) has an universal element $\tilde{L} \in \tilde{I}_E^{sd}$ satisfying the inequality $M(\tilde{L}) < \infty$. By Theorem 2.18 E is a **CSP** - set. \square

Let A and B be subsets of \mathbb{R}^+ . We shall write $A \sqsubseteq B$ if there is $t = t(A, B) > 0$ such that

$$A \cap (0, t) \subseteq B \cap (0, t).$$

The next theorem gives a constructive description of the **CSP** - sets.

Theorem 3.7. *Let $E \subseteq \mathbb{R}^+$. Then E is a **CSP** - set if and only if there are $q > 1$ and a strictly decreasing sequence $\{x_n\}_{n \in \mathbb{N}}, x_n > 0$ for $n \in \mathbb{N}$, such that*

$$\lim_{n \rightarrow \infty} \frac{x_{n+1}}{x_n} = 0 \quad (3.20)$$

and

$$E \sqsubseteq W(q) \quad (3.21)$$

where

$$W(q) := \bigcup_{n \in \mathbb{N}} (q^{-1}x_n, qx_n). \quad (3.22)$$

Proof. The theorem is trivial if $0 \notin acE$. Let us consider the case when $0 \in acE$. Suppose that there are $q > 1$ and a sequence $\{x_n\}_{n \in \mathbb{N}}$ of positive real numbers such that (3.20) and (3.21) hold. Let N_1 and N_2 be natural numbers such that

$$(q^{-1}x_{n+1}, qx_{n+1}) \cap (q^{-1}x_n, qx_n) = \emptyset \quad (3.23)$$

for $n \geq N_1$

$$E \cap (0, t) \subseteq W(q) \cap (0, t) \quad (3.24)$$

for $t \leq x_{N_2}$. Then we have

$$(qx_{n+1}, q^{-1}x_n) \subseteq ExtE$$

for $n \geq N_1 \vee N_2$ and write, in this case, (l_n, m_n) for the unique connected component of $ExtE$ satisfying the inclusion

$$(l_n, m_n) \supseteq (qx_{n+1}, q^{-1}x_n). \quad (3.25)$$

Let $(l_n, m_n) := (l_{N_1 \vee N_2}, m_{N_1 \vee N_2})$ for $n < N_1 \vee N_2$. We claim that $\tilde{L} = \{(l_n, m_n)\}_{n \in \mathbb{N}}$ is universal. Indeed, (3.25) imply that

$$\liminf_{n \rightarrow \infty} \frac{m_n}{l_n} \geq q^{-2} \liminf_{n \rightarrow \infty} \frac{x_{n+1}}{x_n} = \infty.$$

Thus

$$\lim_{n \rightarrow \infty} \frac{m_n}{l_n} = \infty,$$

so that \tilde{L} belongs to \tilde{I}_E^d . Let $\tilde{A} = \{(a_j, b_j)\}_{j \in \mathbb{N}}$ be an arbitrary element of \tilde{I}_E^d . There is $N_3 \in \mathbb{N}$ such that

$$\frac{b_j}{a_j} > q^2 \quad (3.26)$$

and $b_j < (x_{N_1} \vee x_{N_2})$ for $j \geq N_3$. Let $j \geq N_3$. The interval (a_j, b_j) is a connected component of $ExtE$. Consequently, there is $n \geq (N_1 \vee N_2)$ such that either

$$(a_j, b_j) \supseteq (qx_{n+1}, q^{-1}x_n) \quad (3.27)$$

or

$$(a_j, b_j) \subseteq (q^{-1}x_n, x_n). \quad (3.28)$$

Inclusion (3.28) implies

$$\frac{b_j}{a_j} \leq \frac{qx_n}{q^{-1}x_n} = q^2,$$

contrary to (3.26). Hence (3.27) holds. Since for every nonvoid interval $(s, t) \subseteq ExtE$ there is a unique connected component $(a, b) \supseteq (s, t)$, inclusions (3.25) and (3.27) imply the equality $(l_n, m_n) = (a_j, b_j)$. Hence $\tilde{L} \succeq \tilde{A}$ for every $\tilde{A} \in \tilde{I}_E^d$. Thus \tilde{L} is an universal element of (\tilde{I}_E^d, \preceq) .

In accordance with Theorem 2.18 to prove that E is a **CSP** - set it is sufficient to show

$$M(\tilde{L}) = \limsup_{n \rightarrow \infty} \frac{l_n}{m_{n+1}} < \infty. \quad (3.29)$$

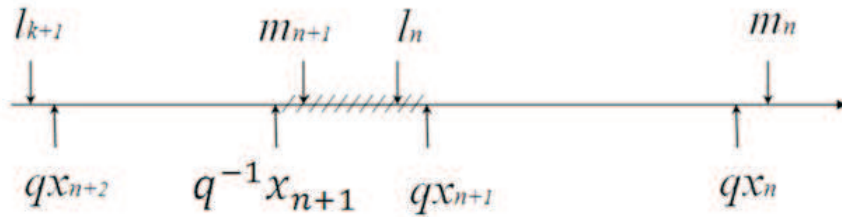


Fig. 3

Since, for sufficiently large n , $(l_n, m_n) \supseteq (qx_{n+1}, q^{-1}x_n)$ and $(l_{n+1}, m_{n+1}) \supseteq (qx_{n+2}, q^{-1}x_{n+1})$ and $l_{n+1} < m_{n+1} < l_n < m_n$ and $qx_{n+2} < q^{-1}x_{n+1} < qx_{n+1} < q^{-1}x_n$ (see Fig. 3), we have

$$m_{n+1}, l_n \in [q^{-1}x_{n+1}, qx_{n+1}].$$

Consequently the inequality

$$\frac{l_n}{m_{n+1}} \leq \frac{qx_{n+1}}{q^{-1}x_{n+1}} = q^2$$

holds for sufficiently large n . Inequality (3.29) follows.

Now assume that E is a **CSP** - set. Let $\{(l_n, m_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E^{sd}$ be universal. Without loss of generality, we may suppose that the sequence $\{l_n\}_{n \in \mathbb{N}}$ is strictly decreasing. Define $\{x_n\}_{n \in \mathbb{N}} := \{m_n\}_{n \in \mathbb{N}}$. Using the inequality $m_{n+1} \leq l_n$ we obtain, from the definition of \tilde{I}_E^d , that

$$\limsup_{n \rightarrow \infty} \frac{x_{n+1}}{x_n} \leq \limsup_{n \rightarrow \infty} \frac{l_n}{m_n} = 0.$$

Thus

$$\lim_{n \rightarrow \infty} \frac{x_{n+1}}{x_n} = 0.$$

To complete the proof it is sufficient to show that there is $q > 1$ such that (3.21) holds. As in the proof of Lemma 2.21, one can easily note that

$$E \setminus \{0\} \subseteq [m_1, \infty) \cup \left(\bigcup_{n \in \mathbb{N}} [m_{n+1}, l_n] \right). \quad (3.30)$$

By formulas (2.19) and (2.20) we have

$$M(\tilde{L}) = \limsup_{n \rightarrow \infty} \frac{l_n}{m_{n+1}} < \infty.$$

Let $q \in (M(\tilde{L}), \infty)$. Then there is $N_4 \in \mathbb{N}$ such that $\frac{l_n}{m_{n+1}} < q$ for $n \geq N_4$. It is clear that $q > 1$. Consequently the inequalities $q^{-1}m_{n+1} < m_{n+1} \leq l_n < qm_{n+1}$ hold for $n \geq N_4$. These inequalities yield the inclusion $[m_{n+1}, l_n] \subseteq (q^{-1}m_{n+1}, qm_{n+1})$. The last inclusion and (3.30) imply

$$E \cap (0, t) \subseteq \left(\bigcup_{n \in \mathbb{N}} (q^{-1}m_n, qm_n) \right) \cap (0, t)$$

for every $t \in (0, m_{N_4+1})$. Relation (3.21) follows. \square

In the case of the closed sets E we may modify Theorem 3.7 by the following way.

Theorem 3.8. *Let $E \subseteq \mathbb{R}^+$ be closed and let $0 \in acE$. Then E is a **CSP** - set if and only if there are $q > 1$ and a strictly decreasing sequence of numbers $x_n > 0$, $n \in \mathbb{N}$, such that $\lim_{n \rightarrow \infty} \frac{x_{n+1}}{x_n} = 0$ and*

$$W(1) \subseteq E \subseteq W(q)$$

where

$$W(a) = \left(\bigcup_{n \in \mathbb{N}} [x_n, ax_n] \right), \quad a \in [1, \infty).$$

The last theorem shows that examples 2.1 and 2.2 give us, in a sense, “the extremal elements” among the closed **CSP** - sets with the accumulation point 0. The proof of Theorem 3.8 is similar to the proof of Theorem 3.7, so we omit it here.

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